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BI-CYCLIC 4-POLYTOPES

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ABSTRACT
In a previous paper [2] we studied the facial structure of convex hulls of certain curves that lie on the torus

$$T^2 = \{(\cos 2\pi x, \sin 2\pi x, \cos 2\pi y, \sin 2\pi y) : |x| \leqq \tfrac{1}{2}, |y| \leqq \tfrac{1}{2}\} \subseteq R^4.$$

In this paper we use the results of [2] to study structure of convex hulls of certain finite subsets of T^2. Specifically, we study the combinatorial structure of the polytopes whose vertex sets are finite subgroups of T^2. Such a subgroup may be represented by Λ/Z^2, where $\Lambda \supseteq Z^2$ is some planar geometric lattice. We shall show how the facial structure of the polytope may be read directly off the lattice Λ. We call these polytopes *bi-cyclic polytopes*; a study of their properties is under preparation.

1. Introduction

In [2] we developed methods which enable the study of convex hulls of subsets of the torus

$$T^2 = \{(\cos 2\pi x, \sin 2\pi x, \cos 2\pi y, \sin 2\pi y) : |x| \leqq \tfrac{1}{2}, |y| \leqq \tfrac{1}{2}\} \subseteq R^4.$$

We used these methods to study the structure of convex hulls of the *generalized trigonometric moment curves* M_{pq}, for any pair p and q of integers:

$$M_{pq} = \{(\cos 2\pi pt, \sin 2\pi pt, \cos 2\pi qt, \sin 2\pi qt) : 0 \leqq t < 1\} \subseteq T^2.$$

It is natural to define a polytope P by taking the convex hull of n evenly spaced points on M_{pq}, and try to determine its facial structure by using similar methods. We consider T^2 as the square of the circle group: $T^2 = R^2/Z^2$, where Z^2 is the integer lattice, and note that the vertices of P form a (translate of a)

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finite cyclic subgroup of T^2. Thus an immediate generalization suggests itself, namely, to consider polytopes whose vertex sets are finite subgroups of T^2, be they cyclic or not. It turns out that the methods of [2] indeed make it possible to determine the combinatorial structure of such polytopes.

The main idea in [2] was as follows. Define a function $\eta : R^2 \to T^2$ by

$$\eta(\theta, \phi) = (\cos 2\pi\theta, \sin 2\pi\theta, \cos 2\pi\phi, \sin 2\pi\phi).$$

If A is a subset of T^2, then $B = \eta^{-1}(A)$ is a (doubly periodic) subset of $R^2 : B = B + Z^2$. For a half-space $H_v^+ = \{u \in R^4 : \langle u, v\rangle \geqq 1\}$ and its bounding hyperplane H_v, define the set

$$\begin{aligned} S(H_v) &= \eta^{-1}(H_v^+) \\ &= \{(\theta, \phi) \in R^2 : \langle(\cos 2\pi\theta, \sin 2\pi\theta, \cos 2\pi\phi, \sin 2\pi\phi), v\rangle \geqq 1\}. \end{aligned}$$

We refer the reader to [2, pages 119–121], where the properties of $S(H_v)$ are thoroughly discussed. Let us just mention that for each $v \in R^4$ there exist nonnegative a, b and real θ_0, ϕ_0 such that

$$\begin{aligned} S(H_v) &= S(a, b, \theta_0, \phi_0) \\ &= \{(\theta, \phi) \in R^2 : a \cos 2\pi(\theta - \theta_0) + b \cos 2\pi(\phi - \phi_0) \geqq 1\}. \end{aligned}$$

Moreover,

$$S(H_v) \neq \varnothing \Leftrightarrow H_v^+ \cap T^2 \neq \varnothing \Leftrightarrow H_v \cap T^2 \neq \varnothing \Leftrightarrow a + b \geqq 1,$$

and

$$\operatorname{bd} S(H_v) = \{(\theta, \phi) \in R^2 : \langle(\cos 2\pi\theta, \sin 2\pi\theta, \cos 2\pi\phi, \sin 2\pi\phi), v\rangle = 1\}.$$

Throughout this paper, E denotes the closed square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. We remark that in the cases we consider the set $S(H_v) \cap E$ is convex, and either it resembles an ellipse, or else it is a strip.

Let A be a subset of T^2. We say that H_v *supports* conv$A$ at some subset A' of A if, and only if, $A \cap H_v^+ = A \cap H_v = A'$. It follows that H_v supports conv$A$ at some subset A' of A if, and only if, the sets $S = S(H_v)$, $B = \eta^{-1}(A)$ and $B' = \eta^{-1}(A')$ satisfy the condition

$$B \cap S = B \cap \operatorname{bd} S = B'. \tag{1}$$

When this happens, we say that S supports B at B'. Note that this definition of support is special to our needs, and is considerably more restrictive than the customary definition. For example, by our definition, a hyperplane that contains the origin cannot be a support hyperplane. As will be shown in the next section, all facets of a non-degenerate bi-cyclic polytope are supported in this restricted sense.

Returning to our special case, A is a subgroup of T^2 and $|A| = n$. Putting $\Lambda = \eta^{-1}(A)$, we find that $A = \Lambda/Z^2$. The planar set Λ is a geometric lattice, so that it may be represented as the set of all linear combinations $ku_1 + lu_2$ of a fixed pair of vectors u_1 and u_2, with integer coefficients. Obviously $\Lambda \supseteq Z^2$, and since $|A| = n$, $\Lambda \subseteq (1/n \cdot Z)^2$. We shall usually be dealing with the lattice Λ, and it will therefore be necessary to take into consideration any identifications modulo Z^2 that may occur.

The lattice Λ/Z^2 has an obvious bi-cyclic structure. We remark without proof that this structure is inherent in the facial structure of the polytope $\Xi = \operatorname{conv} A = \operatorname{conv}\{\eta(\theta, \phi) : (\theta, \phi) \in \Lambda\}$. Hence we call Ξ a *Bi-Cyclic Polytope*. When $(1/n, 2/n) \in \Lambda$, then vert Ξ is a set of evenly spaced points on the curve M_{12}. This is the familiar trigonometric 4-polytope with n vertices (for details, see [1]).

Note that bi-cyclic polytopes are highly symmetric. By [2, §2], for any two vertices v_1 and v_2 of Ξ, there exists an orthogonal automorphism of R^4 which is an automorphism of Ξ and which carries v_1 to v_2. This means that all vertex figures of Ξ are equivalent, up to orthogonal transformation. Moreover, when $(-1, 0, -1, 0) \in \Xi$ (that is, when $(\frac{1}{2}, \frac{1}{2}) \in \Lambda$), then Ξ is centrally symmetric. Another interesting property of bi-cyclic polytopes is that any of their facets (which may be either a simplex, a prism, or an anti-prism) has a dihedral group of orthogonal automorphisms. The order of this group is equal to the number of vertices of the facet. All these symmetries are induced by translation and reflection symmetries of Λ in R^2.

Further discussion of the properties of bi-cyclic polytopes will be found in the concluding section of this paper.

2. The combinatorial structure of Ξ

Our aim in this section is to prove the main theorem of the paper — Proposition 2.2, which provides the connection between the facial structure of Ξ and the structure of the lattice Λ. Throughout this section we let Ξ be a fixed bi-cyclic polytope with n vertices and $\Lambda = \eta^{-1}(\text{vert } \Xi)$ the associated lattice.

DEFINITION 2.1. (a) A *Face Parallelogram* (F.P.) of Λ is a fundamental parallelogram of Λ (i.e. a parallelogram whose area is $1/n$), with four distinct vertices modulo Z^2, such that two of its edges have positive slope, the other two negative slope. If two or more of the vertices are equal modulo Z^2, then the parallelogram will be called a *pseudo-F.P.*

(b) A (vertical or horizontal) *Elementary Line* (E.L.) of Λ is a (vertical or horizontal) line that contains at least two points of Λ that are distinct modulo Z^2.

(c) An *Elementary Vertical Strip* (E.V.S.) of Λ is a closed vertical strip in the plane, bounded by two distinct E.L.'s, with minimal width.

(d) An *Elementary Vertical Strip* (E.V.S.) of Λ is similarly defined.

(e) An *Elementary Triangle* (E.T.) of Λ is any triangle of Λ that can be completed, by adjoining another vertex of Λ, to form an F.P.

REMARK. All these sets are considered as subsets of Λ, i.e., an F.P. is a set of four vertices of Λ, whose convex hull is a parallelogram which satisfies the prescribed conditions. The reader may have noted that the faces of Ξ do not correspond to subsets of Λ, or even of E, since they are not subsets of T^2. Thus the correspondence is between the vertex sets, and we draw the parallels between Λ and Ξ using condition (1) above. In this way we shall feel free to use terminology which is mixed between Λ and Ξ, as for example in saying that some E.V.S. is an antiprism. Hopefully, this will aid readability, perhaps at the expense of meticulousness.

It will be convenient at this stage to discard three special cases. First, the case $\Lambda = \{(i/n, i/n), 0 \leqq i < n\} + Z^2$ or $\Lambda = \{(i/n, -i/n), 0 \leqq i < n\} + Z^2$. In these cases Ξ is a 2-dimensional n-gon. Second, the case $\Lambda = (1/k \cdot Z) \oplus (1/l \cdot Z)$, with $kl = n$. In this case we say that Λ is *rectangular*. Note that Λ is rectangular if and only if Λ has a fundamental parallelogram whose edges are parallel to the axes of R^2. It now follows, directly from the definition of Ξ, that Ξ is the direct product of a k-gon and an l-gon, and so the structure of Ξ is completely determined. Last, we discard all low-dimensional lattices that were not thrown away already. These may be of two types: those with at most 4 vertices, and those for which Ξ is a 3-antiprism over an $n/2$-gon. This last case occurs when Λ contains a subgroup with index 2 which lies on one of the coordinate axes. The reader is invited to verify that in all other cases, Λ properly contains either an F.P., or an E.V.S., or an E.H.S. (One way of verifying this is to check that the claim holds for all lattices that contain an element with order at least 5, and for all lattices with $n = 6$, $n = 8$ and $n = 9$ elements. It follows that it holds for all $n > 4$.) Proposition 2.2 then implies that Ξ has a proper facet, so that it is full dimensional.

We call any of these lattices *degenerate*, and all others *non-degenerate*. For the rest of this paper, we shall assume that Λ is a non-degenerate lattice.

The following proposition gives a full description of the facial structure of Λ. It lists its facets and 2-faces, and its proof will occupy the rest of the paper.

PROPOSITION 2.2. *Suppose that* Ξ *is a non-degenerate bi-cyclic polytope. Then*

(a) *The facets of* Ξ *are precisely all F.P.'s, E.V.S.'s and E.H.S.'s.*

(b) *The 2-faces of* Ξ *are precisely all E.T.'s, and all E.L.'s that contain three or more distinct points modulo* Z^2.

(c) *F.P.'s are tetrahedra. E.V.S.'s are antiprisms over an E.L., which is a polygon or an edge. When the base is an edge, the antiprisms become tetrahedra. E.T.'s are triangular 2-faces of* Ξ.

The major tool that will be employed in the proof is corollary 7 of [2], which we now reproduce as Lemma 2.3. Recall that E denotes the closed square $[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]$.

LEMMA 2.3. *Let* $a \geqq 0$, $b \geqq 0$, $a+b>0$ *and* $S=S(a,b,0,0)$ *be given. Suppose that* $L \subseteq R^2$ *is a line which is not coincident with a coordinate axis. Suppose further that L cuts both axes in E and at least one axis in* int E. *Then*

(a) $|E \cap L \cap \operatorname{bd} S| \leqq 2$.

(b) *If* $\{E \cap L \cap \operatorname{bd} S\} = \{u, v\}$, *then* $]u,v[\ \subseteq \operatorname{int} S$, *and* $E \cap L \cap S = [u, v]$.

(c) *If* $a>0$ *and* $b>0$ *and L is tangent to* bd S *at a point* $v \in \operatorname{int} E$, *then* $E \cap L \cap S = E \cap L \cap \operatorname{bd} S = \{v\}$. □

Proposition 2.2 will be proved in the following steps.

S1. Let B' denote the centrally symmetric translate of an F.P., B, and suppose that L is a line which contains an edge of B'. Then

(a) L satisfies the conditions of Lemma 2.3.

(b) For a vertex (x,y) of B', $|x|+|y|<\frac{1}{2}$ and $\cos 2\pi x+\cos 2\pi y>0$.

S2. An F.P. is a face of Λ. (This is the main step.)

S3. E.V.S.'s and E.H.S.'s are faces of Λ; E.T.'s are faces, and therefore 2-faces. It follows that F.P.'s are 3-faces, and hence that they are tetrahedra.

S4. E.L.'s are k-gons. When $k=2$, they degenerate into edges. E.V.S.'s and E.H.S.'s are antiprisms whose bases are E.L.'s and whose side facets are E.T.'s.

S5. There are no other facets (3-faces) or subfacets (2-faces).

PROOF OF **S1.** (a) We have to show that if L_1 and L_2 are lines that contain parallel edges of B', then both the horizontal and vertical distances between L_1

and L_2 are less than or equal to 1, with at least one of the inequalities sharp. The first part follows from the following observation: since B is a fundamental parallelogram of Λ, the open strip between L_1 and L_2 does not contain any point of Λ, while on the other hand $\Lambda \supseteq Z^2$. For the second part, if both distances are 1, then the slope of these lines must be either $+1$ or -1; but this would mean that Λ is degenerate, contrary to our assumptions.

(b) From part (a), $B' \subseteq \operatorname{conv}\{\pm(0, \frac{1}{2}), \pm(\frac{1}{2}, 0)\}$. It follows that $|x| + |y| \leqq \frac{1}{2}$. Also, since B contains four distinct points modulo Z^2, $(x, y) \neq (0, \pm\frac{1}{2})$ and $(x, y) \neq (\pm\frac{1}{2}, 0)$. Thus, if $|x| + |y| = \frac{1}{2}$, any line through (x, y) cuts both coordinate axes outside $\operatorname{int} E$ or one of the axes outside E, contradicting (a). Last, since $0 \leqq |x| < \frac{1}{2} - |y| \leqq \frac{1}{2}$, we have $\cos 2\pi x > \cos 2\pi(\frac{1}{2} - |y|) = -\cos 2\pi y$.

Remark. These results show that an F.P. is always contained in a translate of $\operatorname{int}\operatorname{conv}\{\pm(0, \frac{1}{2}), \pm(\frac{1}{2}, 0)\}$.

Proof of **S2**. For an F.P., B, we must find $a \geqq 0$, $b \geqq 0$, θ_0 and ϕ_0, such that $S = S(a, b, \theta_0, \phi_0)$ satisfies condition (1):

$$\Lambda \cap S = \Lambda \cap \operatorname{bd} S = B + Z^2.$$

Let (θ_0, ϕ_0) denote the center of B. Put $B' = B - (\theta_0, \phi_0)$, $\Lambda' = \Lambda - (\theta_0, \phi_0)$. It suffices to find $a \geqq 0$ and $b \geqq 0$ such that $S' = S(a, b, 0, 0)$ satisfies

$$\Lambda' \cap S' = \Lambda' \cap \operatorname{bd} S' = B' + Z^2.$$

We shall show this in a somewhat devious manner, exploiting the properties of S which were stated in Lemma 2.3.

Suppose then that B' is the centrally symmetric translate of B, and that $\operatorname{vert} B' = \{\pm(x_1, y_1), \pm(x_2, y_2)\}$.

I. First, we shall show that there exist $a > 0$ and $b > 0$ such that

$$(2) \qquad \begin{aligned} a\cos 2\pi x_1 + b\cos 2\pi y_1 &= 1, \\ a\cos 2\pi x_2 + b\cos 2\pi y_2 &> 1. \end{aligned}$$

By using the symmetries of (2), we may assume that (x_1, y_1) is the rightmost vertex of B', that (x_2, y_2) is the top vertex, and that $y_1 \geqq 0$ (see Fig. 1).

Define $T = \operatorname{conv}\{(0, \frac{1}{2}), (x_1, y_1), -(x_1, y_1)\}$. We claim that $(x_2, y_2) \in T \setminus \operatorname{vert} T$; for otherwise, either the line through (x_1, y_1) and (x_2, y_2), or the line through $-(x_1, y_1)$ and (x_2, y_2) would meet the y-axis outside E, contrary to **S1**(a).

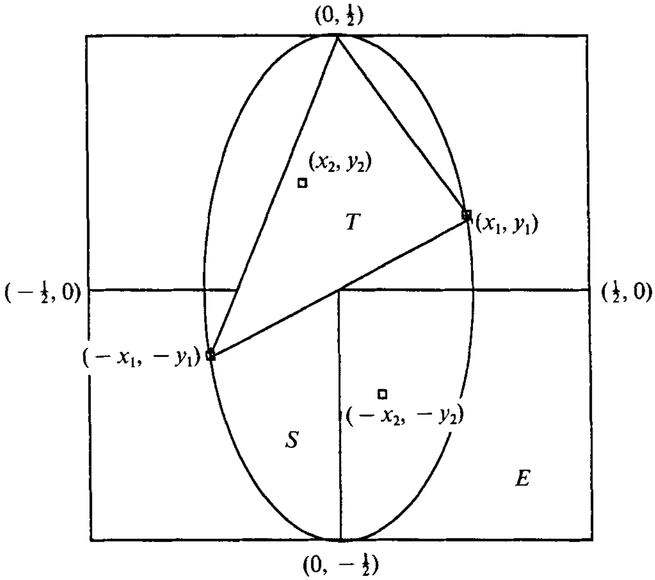

Fig. 1.

Let a and b denote the solutions of the equation system

$$\begin{aligned} a\cos 2\pi x_1 + b\cos 2\pi y_1 &= 1, \\ a \qquad - b \qquad &= 1. \end{aligned}$$

(This is equivalent to replacing (x_2, y_2) by $(0, \frac{1}{2})$ in (2).) We have

$$a = \frac{1 + \cos 2\pi y_1}{\cos 2\pi x_1 + \cos 2\pi y_1}, \qquad b = \frac{1 - \cos 2\pi x_1}{\cos 2\pi x_1 + \cos 2\pi y_1}.$$

By **S1**(b), $\cos 2\pi x_1 + \cos 2\pi y_1 > 0$. Also, $|x_1| + |y_1| < \frac{1}{2}$, and with our assumptions $0 < x_1 < \frac{1}{2}$ and $0 \leqq y_1 < \frac{1}{2}$. It follows that a and b are positive, as required.

It remains to show that a and b satisfy $a\cos 2\pi x_2 + b\cos 2\pi y_2 > 1$. Put $S = S(a, b, 0, 0)$. The points $(0, \frac{1}{2})$ and (x_1, y_1) lie on bd S, and Lemma 2.3 is applicable to the line L through them. Thus, the open interval $](0, \frac{1}{2}), (x_1, y_1)[$ lies in int S. A similar argument holds for $-(x_1, y_1)$. It follows that $T \setminus \operatorname{vert} T \subseteq \operatorname{int} S$ (cf. [2, Lemma 3b]), and in particular that $(x_2, y_2) \in \operatorname{int} S$. By definition, this means that

$$a\cos 2\pi x_2 + b\cos 2\pi y_2 > 1,$$

as required.

II. We shall now find $a \geqq 0$ and $b \geqq 0$ such that $S' = S(a, b, 0, 0)$ satisfies

$$\Lambda' \cap S' \supseteq \Lambda' \cap \operatorname{bd} S' \supseteq B' + Z^2.$$

In other words, we shall show that both x_1, y_1 and x_2, y_2 lie on bd S'. This is equivalent to the condition

$$\begin{aligned} a \cos 2\pi x_1 + b \cos 2\pi y_1 &= 1, \\ a \cos 2\pi x_2 + b \cos 2\pi y_2 &= 1. \end{aligned} \tag{3}$$

Using (2) twice, we obtain positive a_1, b_1, a_2 and b_2 such that

$$a_1 \cos 2\pi x_1 + b_1 \cos 2\pi y_1 = 1, \qquad a_2 \cos 2\pi x_1 + b_2 \cos 2\pi y_1 = d > 1,$$

$$a_1 \cos 2\pi x_2 + b_1 \cos 2\pi\lambda y_2 = c > 1, \qquad a_2 \cos 2\pi x_2 + b_2 \cos 2\pi y_2 = 1.$$

Defining $\alpha = (d-1)/(dc-1)$, $\beta = (c-1)/(dc-1)$, we see that

$$a = \alpha a_1 + \beta a_2$$

and

$$b = \alpha b_1 + \beta b_2$$

are the solutions of (3), as required.

III. It remains to show that for these a and b, $S' = S(a, b, 0, 0)$ contains no points of Λ' other than $\pm (x_1, y_1)$ and $\pm (x_2, y_2)$. To show this, consider the four lines L_1, L_2, L_3 and L_4 that contain the edges of B' (see Fig. 2). The two

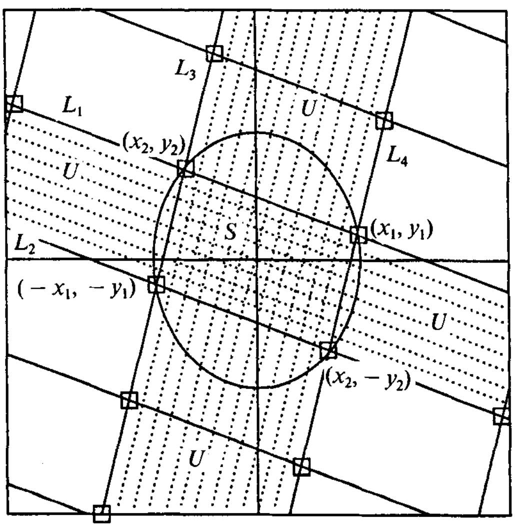

Fig. 2.

open strips, bounded by the two pairs L_1 and L_2, L_3 and L_4 of parallel lines, do not contain any point of Λ. Denote the union of these two open strips by U. We will show that $E \cap S' \setminus \{\pm(x_1, y_1), \pm(x_2, y_2)\}$ is contained in U; this will complete the argument.

Suppose then that our claim does not hold, that is, suppose that a point $z \in E \cap S'$ exists, with $z \notin U$. If z lies on one of the lines L_1, L_2, L_3 or L_4, then Lemma 2.3(b) may be applied to that line, and a vertex of B' found in int S'; but this contradicts (3). Suppose now that z does not belong to any of these lines: since U is starshaped with respect to the origin, and since $S' \cap E$ is strongly starshaped with respect to the origin ([2, Lemma 3]), it follows that $z' =]0, z[\cap \operatorname{bd} U$ satisfies $z' \in \operatorname{int} S'$. If $z' \in \operatorname{vert} B'$, this is a contradiction as above. Otherwise, z' lies on the boundary of U and we are back in the situation of the first case.

Thus $\Lambda' \cap S' = \Lambda' \cap \operatorname{bd} S' = B'$.

PROOF OF S3. We have to show (a) that E.V.S.'s and E.H.S.'s are faces, (b) that E.T.'s are faces.

Claim (a) can be checked directly from the definitions of Ξ and Λ. Nevertheless we give a proof using our methods. Suppose that D is an E.V.S. bounded by the lines $x = x_1$ and $x = x_2$. Put $a = 1/(\cos \pi(x_2 - x_1))$, $b = 0$, $\theta_0 = (x_1 + x_2)/2$, $\phi_0 = 0$, and $S = S(a, b, \theta_0, \phi_0)$. We now obtain $S = \{(x, y) \in R^2 : |x - \theta_0| \leqq |x_2 - x_1|/2\}$, and $\Lambda \cap S = \Lambda \cap \operatorname{bd} S = D$.

(b) We shall now show that an E.T. is a face of Λ. We do this by showing that if Δ is an E.T. of Λ then there are faces F_1 and F_2 of Λ, $F_1 \not\equiv F_2 (\operatorname{mod} Z^2)$, such that $\Delta = F_1 \cap F_2$.

We start by finding F_1 and F_2. By definition, there is an F.P. that contains Δ. Name this F.P. F_1. If Δ contains a vertical or a horizontal edge, then F_2 is an E.V.S. or an E.H.S., respectively. Otherwise, there are precisely three ways to complete Δ to form a fundamental parallelogram (choose which edge is to become the diagonal). Precisely two of these parallelograms, F_1 and F_2, satisfy the edge slope condition of Definition 2.1(a). If both are F.P.'s, we are done. The other possibility is that F_2 is a pseudo-F.P.

Suppose then, to be concrete, that $\operatorname{vert} F_1 = \{a, b, c, d\}$, that $\operatorname{vert} F_2 = \{b, c, d, e\}$, and that $\operatorname{vert} \Delta = \{b, c, d\}$ (see Fig. 3). Since F_1 is an F.P., b, c and d are all distinct. Checking the possibilities for congruence modulo Z^2, we find that $e \equiv d (\operatorname{mod} Z^2)$ or $e \equiv c (\operatorname{mod} Z^2)$ contradicts the fact the F_1 is an F.P. The remaining equivalence is $e \equiv b (\operatorname{mod} Z^2)$. Thus, $\{b, c, d\} \equiv \{e, c, d\} (\operatorname{mod} Z^2) = \operatorname{vert} \Delta$; taking $f = d + (d - b)$ and redefining $F_2 =$

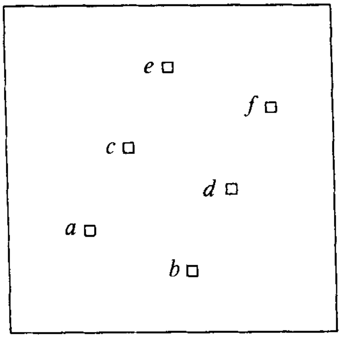

Fig. 3.

$\{c, d, e, f\}$, we find that $\Delta = F_1 \cap F_2$. Moreover, F_2 is an F.P., being a translate of F_1.

It remains to show that $F_1 \not\equiv F_2 (\mathrm{mod}\, Z^2)$. Using arguments similar to those used in the preceding paragraph, we see that $F_1 \equiv F_2 (\mathrm{mod}\, Z^2)$ implies both $a \equiv f (\mathrm{mod}\, Z^2)$ and $b \equiv e (\mathrm{mod}\, Z^2)$. The complete lattice is generated by $u = c - a$ and $v = b - a$, and the congruencies imply the relations $2u \equiv v$ and $v + 2u \equiv 2v \equiv 0$. It follows that $\Lambda = \{a, b, c, d\} + Z^2$ so that the lattice is degenerate. This completes the proof of **S3**.

PROOF OF **S4**. This can be proved directly from the definition of η. Details are left to the reader.

PROOF OF **S5**. To show that our list of facets and subfacets is complete, we employ an argument due to M.A. Perles (unpublished). The argument is interesting in its own right and has proved useful in several other cases:

LEMMA 2.4 (M.A. Perles). *Suppose that $\mathscr{F}$ is a nonempty list of facets of a polytope P. If, for every facet F in $\mathscr{F}$, and for every facet G of F, G is contained in two distinct members of $\mathscr{F}$, then $\mathscr{F}$ is the complete list of facets of P.*

The proof is left to the reader. □

Our required result is a direct consequence, since we have shown (in **S3**) that every E.T. is contained in two facets, and certainly so is every E.L.

Thus Proposition 2.2 is proved. □

3. Conclusion

We have seen how the facial structure of a bicyclic polytope P can be read off its lattice representation Λ. Although this does not exhibit the combinatorial stucture in closed form, it does make it easy to find the structure of any particular bi-cyclic polytope, and to discover some general properties of these polytopes. For example, a bi-cyclic polytope is simplicial if, and only if, Λ has no facets with more than 4 vertices, or, in other words, no E.L.'s with more than 2 distinct points. This happens precisely when each of the intersections of $\Lambda \cap E$, with the x-axis and with the y-axis, contains at most two points. One implication is that if $|\text{vert } \Xi| = p$, a prime number, then Ξ must be simplicial.

An important problem is that of finding all F.P.'s of a given lattice. The regularity of Ξ means that it suffices to find the set of all F.P.'s that contain a given vertex, say (0, 0). Assume that $|\text{vert } \Xi| = n$. The symmetry of the vertex figure means that we need only find numbers i, j, k, l such that

(1) $(i/n, j/n) \in \Lambda$, $(k/n, l/n) \in \Lambda$.

(2) $ik + jl = n$.

(3) $0 < i, j, k < n$, $-n < l < 0$.

This provides a practical method for finding the structure of any given bi-cyclic polytope.

An ongoing research ([3]) provides some additional information. The most interesting results are:

(1) The lattice Λ can be reconstructed from the combinatorial structure of Ξ (up to the eight-fold symmetries of the axes of the plane). Thus Λ is an invariant of the combinatorial type of Ξ.

(2) The number of combinatorial types of bi-cyclic polytopes with n vertices is at least $[n/4]$, with equality holding when $n = p$, a prime number.

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